Lezione 14

FORME DIFFERENZIALI

M n-varietà orientata $\qquad \omega \in \Omega^n_c(M)$

$$\int_M \omega \in \mathbb{R} \qquad \omega = \sum \omega_i$$

Oss: $B \subseteq M$ Boreliano $\int_B \omega$ stessa def.

$$B = \bigsqcup B_i \Rightarrow \int_B \omega = \sum \int_{B_i} \omega$$

Es: $T = \underbrace{S^1 \times \dots \times S^1}_{n}$.

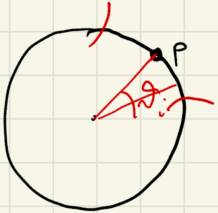
$$\vartheta = (\overset{\circlearrowleft}{\vartheta_1}, \dots\dots, \vartheta_n) \qquad \vartheta_i \in \mathbb{R}/_{2\pi\mathbb{Z}} \qquad S^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$$

$\vartheta_1 \pm 2k\pi$

S^1:

$d\vartheta_i$ = differenziale della "funzione" ϑ_i

funzioni locali unica a meno di aggiungere $2k\pi$



$d\vartheta_i$ è definito globalmente

$$d\vartheta_1 \in \Omega^1(S^1)$$

$$\underline{\omega = d\vartheta_1 \wedge \dots \wedge d\vartheta_n}$$

$d\vartheta$ su S^1

$d\vartheta_i$ sul S^1 i-esimo

$M \times N$

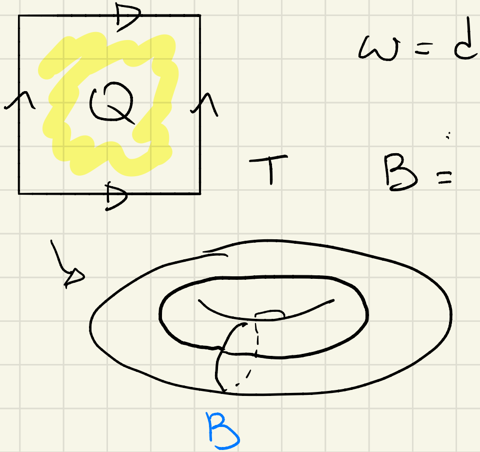
$\eta_1 \in \Omega^k(M)$

$\eta_2 \in \Omega^h(N)$

$$\underset{\pi^*(\eta_1)}{\underset{\parallel}{\eta_1}} \wedge \underset{\pi^*(\eta_2)}{\underset{\parallel}{\eta_2}} \in \Omega^{k+h}(M \times N)$$

$$\begin{tikzcd} M \times N \arrow[r, "\pi"] \arrow[d, "\pi"] & N \arrow[l, bend right] \\ M & \end{tikzcd}$$

Calcolare $\int_T \omega$



$\omega = d\vartheta_1 \wedge d\vartheta_2$

$B := \pi(\partial Q) \subseteq T$

Oss: Se B ha misura nulla allora $\int_B \omega = 0$

$$\int_T \omega = \underbrace{\int_B \omega}_{\overset{\shortparallel}{0}} + \underbrace{\int_{T\setminus B} \omega}_{U}$$

$$\int_U \omega = \int_{\varphi(U)} \varphi_* \omega$$

$\varphi : U \xrightarrow{id} \mathbb{R}^n$

$\vartheta_i \longmapsto x_i$

$Q = [0,2\pi]^n$

$T = Q/\sim$

$U = (0,2\pi)^n$

$$= \int_{(0,2\pi)^n} 1 \cdot dx^1 \wedge \_\_ \wedge dx^n = \int_{(0,2\pi)^n} 1 = \boxed{(2\pi)^n}$$

SOTTOVARIETA'

$S \underset{cpt}{\subseteq} M^n$ S sottovarietà k-dim orientata $\omega \in \Omega^k_{\cancel{c}}(M)$

Def: $\int_S \omega := \int_S i^*(\omega)$ $\qquad i : S \hookrightarrow M$

FORMA VOLUME

M^n orientata

Def: Una FORMA VOLUME è una $\omega \in \Omega^n(M)$ t.c.

$\forall p \qquad \boxed{\underline{\omega(p)(v_1, \ldots, v_n) > 0}}$ ⊛ $\qquad \forall \underbrace{v_1, \ldots, v_n}_{\text{base } +} \in T_pM$

Se ω è forma volume, dato $B \subseteq M$ Boreliano

Def: $\mathrm{Vol}(B) := \int_B \omega$ VOLUME DI B

Oss: $\underline{\mathrm{Vol}(B) \geq 0}$ e se $\mathring{B} \neq \emptyset$ allora $\underline{\mathrm{Vol}(B) > 0}$

perché è così in $\mathbb{R}^n$ $\qquad \omega = \underbrace{f(x)\, dx^1 \wedge \ldots \wedge dx^n}$ trasportata lungo carte +

✪ ω forma volume $\Longleftrightarrow f(x)>0 \ \forall x$

Ottengo una MISURA su M

Oss: Se M orientata ha un tensore metrico g allora g induce una forma volume ω

segnatura arbitraria

(struttura pseudo-Riem.)

ω è caratterizzata da queste richieste:

$$\omega(p)(\underbrace{v_1,\dots,v_n}_{\text{ortonormale positiva}}) = 1 \qquad g(v_i,v_i) = \pm 1$$

g Lorentziana $\Longleftrightarrow$ segnatura $(n-1,1)$

È ben def perché se $w_1,\dots,w_n$ orton positiva $\Rightarrow$ matrice di cambio base da v_i a w_i ha det $\boxed{1}$

$${}^tAA = I \qquad SO(n)$$

$$\Rightarrow \det A = 1$$

$$\underbrace{\mathbb{R} = \Lambda^n V} \qquad \dim V = n$$

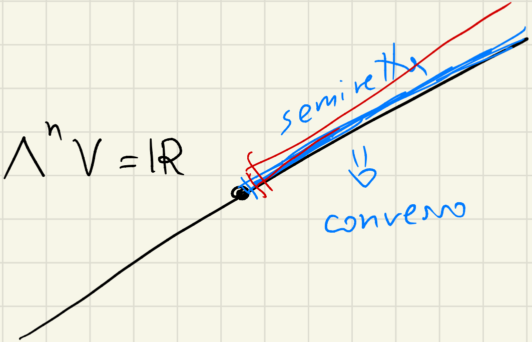
Prop: Ogni M orientata ha forma volume

dim: 1) Ogni M ha g Riem. $\Rightarrow$ forma volume

2) $\{\varphi_i : U_i \to \mathbb{R}^n\}$ orientato ρ_i

su U_i $\sum \varphi_i^*(dx^1 \wedge \dots \wedge dx^n) \cdot \rho_i$ $\Lambda^n V = \mathbb{R}$

stessa dimo di $\exists\, g$ Riem.



Oss: Se $\omega \in \Omega^n(M)$ forma volume, $f \in \mathcal{C}^\infty(M)$ $f(x) > 0 \ \forall x \in M$

allora $\omega' = f\omega$ è forma volume

Qualsiasi forma volume ω' è di questo tipo $\omega' = f\omega$
$f(x) > 0$

DERIVATA ESTERNA

M^n varietà

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] \arrow[d, phantom, "\shortparallel"] & \Omega^1(M) \arrow[r, "d"] \arrow[d, phantom, "\shortparallel"] & \Omega^2(M) \arrow[r, "d"] & \cdots \\
\mathcal{C}^\infty(M) & \Gamma T^*(M) & &
\end{tikzcd}$$

Def in $\mathbb{R}^n$:

$U \subseteq \mathbb{R}^n$ $\qquad$ $\omega = \sum_I f_I \, dx^I$ $\qquad$ I multi-indici

$f_I \wedge dx^I$

$I = (i_1, \dots, i_k)$

$i_1 < \dots < i_k$

$\omega \in \Omega^k(U)$

$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$d\omega = \sum_I df_I \wedge dx^I$$

Prop: La definizione è ben posta (non dip. da carta) e induce

1) $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ lineare, che ha queste proprietà:

$\forall \omega \in \Omega^k(M)$
$\eta \in \Omega^h(M)$

2) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

3) $d \circ d = 0$

dim: Dimostriam 1), 2) 3) in carta:

2) $\omega = \sum_I f_I \, dx^I \qquad \eta = \sum_J g_J \, dx^J$

per linearità
$\omega = f_I \, dx^I$
$\eta = g_J \, dx^J$

$$d(\omega \wedge \eta) = d\left(f_I dx^I \wedge g_J dx^J\right)$$

$$= d\left(f_I g_J \, dx^I \wedge dx^J\right)$$

$$= d(f_I g_J) \wedge dx^I \wedge dx^J$$

$$= df_I \wedge g_J \wedge dx^I \wedge dx^J + f_I \wedge \underbrace{dg_J}_{} \wedge \overset{k}{\boxed{dx^I}} \wedge dx^J$$

$$= \left(df_I \wedge dx^I\right) \wedge \left(g_J \wedge dx^J\right) + (-1)^k f_I \wedge dx^I \wedge dg_J \wedge dx^J$$

$$d\omega \wedge \eta \qquad\qquad (-1)^k \omega \wedge d\eta$$

3) $d \circ d = 0$ $\qquad \omega = f_I dx^I = f dx^I$

$$d\omega = df \wedge dx^I = \boxed{\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I}$$

$$d(d\omega) = \sum_{\substack{i=1 \\ j=1}}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} \underbrace{dx^j \wedge dx^i}_{} \wedge dx^I = 0 \qquad dx^i \wedge dx^i = 0$$

$i \neq j$ simm. antis.

$d : \Omega^k(U) \to \Omega^{k+1}(U)$ lineare è caratterizzata dalle

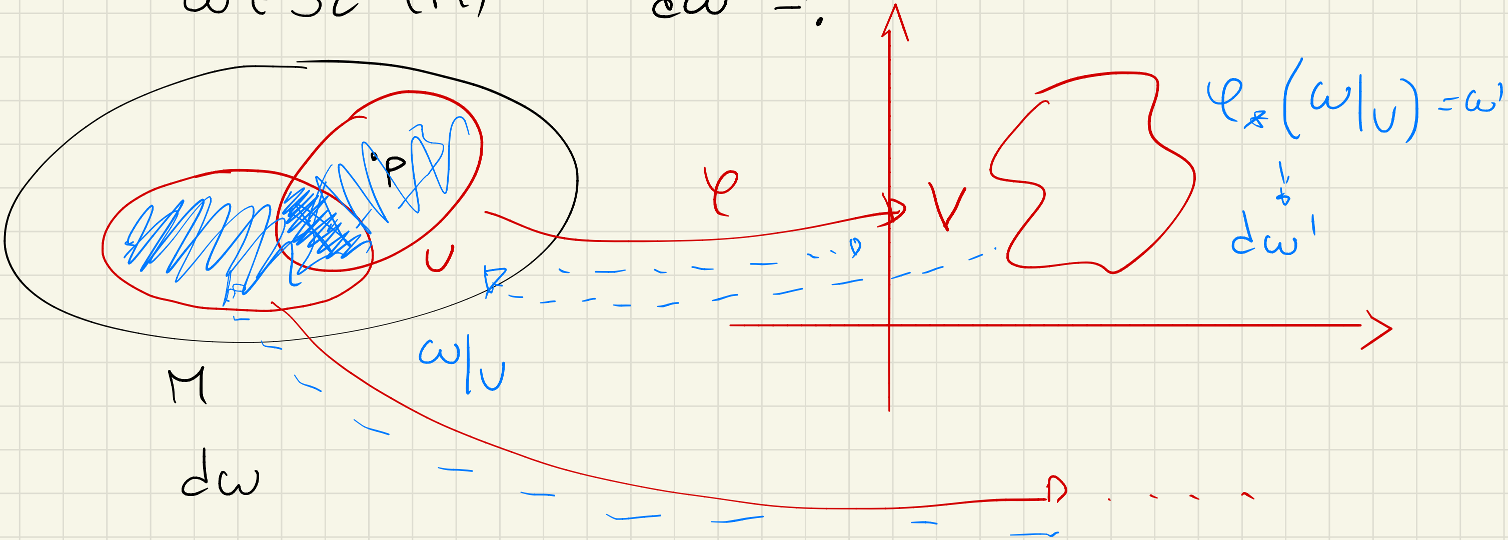
1), 2), 3) + 0) df è il differenziale (esercizio)

FATELO!

$$\omega = \sum_I f_I \, dx^I \qquad d\omega$$

Quindi è ben definito anche su M !

$$\omega \in \Omega^k(M) \qquad d\omega = ?$$



Teorema di Stokes

M

$\omega \in \Omega^k(M)$

$$\boxed{\begin{array}{l} \partial\partial M = \emptyset \\ dd\omega = 0 \end{array}}$$

$i : \partial M \to M \qquad \int_{\partial M} \omega = \int_{\partial M} i^*(\omega)$

dim :

Teo : M^{n+1} orientata $\quad \omega \in \Omega^n_c(M)$

$$\boxed{\int_M d\omega = \int_{\partial M} \omega} \quad \longleftarrow$$

$$\underset{\text{ori}}{\uparrow} \; \dashrightarrow \; \underset{\text{ori}}{\uparrow}$$

Caso $M = \mathbb{R}^{n+1}_+$ semispazio $x_{n+1} \geq 0$

$\omega \in \Omega^n_c(M) \qquad \omega = \sum_{i=1}^{n+1} \omega_i$

$$\omega_i = f_i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1}$$

$$d\omega_i = \sum \frac{\partial f_i}{\partial x_j} \underline{dx^j} \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1}$$

$$\int_M \eta = -\int_{-M} \eta$$

$\eta \in \Omega^n(M)$

$$= \frac{\partial f_i}{\partial x_i}\, dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$$

$$= (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\, dx^1 \wedge \dots \dots \wedge dx^{n+1} \quad \leftarrow$$

$\boxed{\underline{i \leq n :}}$

$$\int_{\mathbb{R}^{n+1}_+} d\omega_i = (-1)^{i-1} \int_{\mathbb{R}^{n+1}_+} \frac{\partial f_i}{\partial x_i}\, dx^1 \dots dx^{n+1} = \int_{\mathbb{R}^n_+} \underbrace{\left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}\, dx^i \right)}_{\overset{\shortparallel}{0}} dx^1 - \widehat{dx^i} - dx^{n+1}$$

$$\mathbb{R}^{n+1}_+ = \underset{x^i}{\mathbb{R}} \times \mathbb{R}^n_+$$

$$\lim_{T\to\infty} \Big[f_i \Big]_{-T}^{T} = \lim_{T\to\infty} f_i(T) - f_i(-T) \overset{}{=} \overset{\shortparallel}{0}$$

$$\rightarrow \int_{\partial \mathbb{R}^{n+1}_+} \omega_i =$$

$$\omega_i = f_i\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \boxed{dx^{n+1}}$$

$$\int_{\partial \mathbb{R}^{n+1}_+} i^*(\omega) = 0$$

$$i^*(\omega_i) = f\, i^*(dx^1) \wedge \dots \dots \wedge \dots \wedge \dots \wedge \underbrace{i^*(dx^{n+1})}_{\overset{\shortparallel}{0}}$$

$$i^*(dx^i) = dx^i \quad i \leq n$$

$$\underline{i^*(dx^{n+1}) = 0} \qquad \underline{\text{ex}}$$

$\underline{i=n+1}$

$$\int_{\mathbb{R}^{n+1}_+} d\omega_{n+1} = (-1)^n \int_{\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n} \frac{\partial f_{n+1}}{\partial x_{n+1}} dx^1 \cdots \cdots dx^{n+1} \overset{(-1)^n}{=} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}_+} \frac{\partial f_{n+1}}{\partial x_{n+1}} dx^{n+1} \right) dx^1 \cdots dx^n$$

$$= (-1)^n \int_{\mathbb{R}^n} \Big[f_{n+1}(x) \Big]_0^{\overset{x_{n+1}}{\infty}} dx^1 \text{—} dx^n = \boxed{(-1)^{n+1} \int_{\mathbb{R}^n} f_{n+1}(x)\, dx^1 \text{—} dx^n}$$

$$\Big[f_{n+1}(x) \Big]_0^{\infty} = f_{n+1}(x_1, \text{—}, x_n, \infty) - f_{n+1}(x_1, \text{—}, x_n, 0)$$

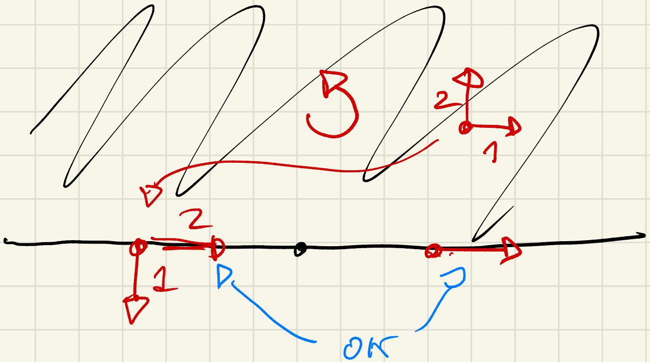
$f_{n+1}(x_1, \text{—}, x_n, \infty) \leftarrow 0$

$x = (x_1, \text{—}, x_n, 0)$

$$\int_{\partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n} \omega_{n+1} = \boxed{\int_{\partial \mathbb{R}^{n+1}_+} f_{n+1}(x_1, \text{—}, x_n, 0)\, dx^1 \wedge \text{——} \wedge dx^n}$$

L'orientazione su $\partial \mathbb{R}^{n+1}_+$ è indotta da quella di $\mathbb{R}^{n+1}_+$

coincide con quella di $\mathbb{R}^n \iff n$ dispari



$\mathbb{R}^2$

$\boxed{ex}$

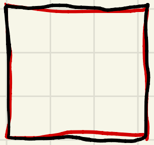
$(x_1, \_, x_n)$

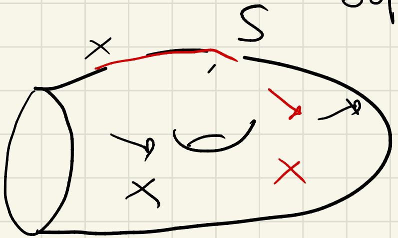
$(x_2, \_, x_n, x_1)$

$M = \mathbb{R}^n_+$

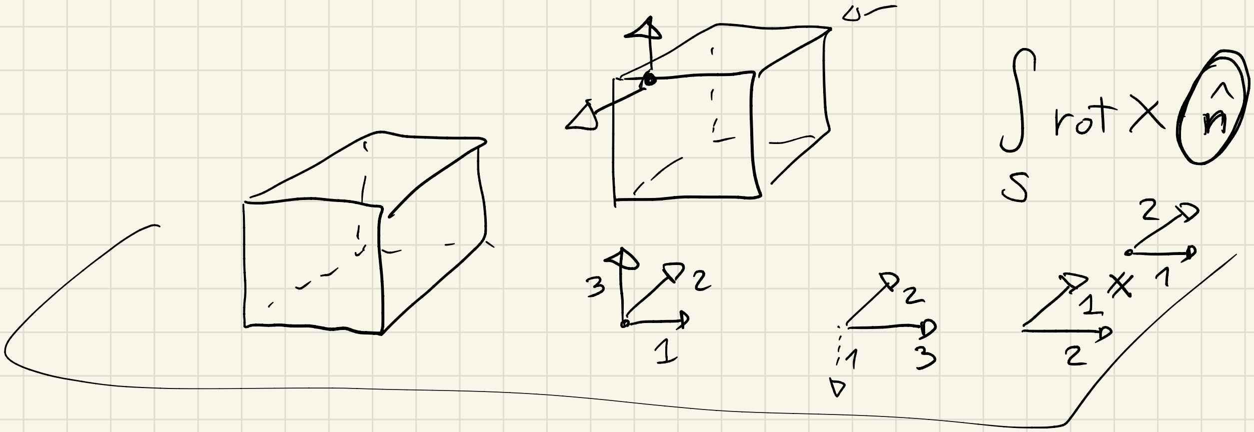
M^{n+1} generale $\quad \omega \in \Omega^n_c(M) \quad \omega = \sum \omega_i$ OK

supp $\omega_i \subseteq U \xrightarrow{\varphi} \mathbb{R}^n_+$





$$\int_S \text{rot} X \cdot \hat{n} = \int_{\partial S} X \cdot \hat{t}$$



$$\int_S \text{rot}\, X \cdot \boxed{\hat{n}} = 0$$